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$su(1,1)$ algebraic description of one-dimensional potentials within the R-matrix theory

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Abstract. The eigenstates of a particle in a rectangular-well potential with appropriate boundary conditions are proved to be the standard basis of an irreducible representation of the $su(1,1)$ Lie algebra. The algebra operators are constructed explicitly and the energy levels and the R-function are calculated. Due to the general connection between the generators of $su(1,1)$ we can algebraically relate a wide class of one-dimensional potentials to the $su(1,1)$ Lie algebra in this framework. This algebraic approach allows us to write an algebraic parametrization for the R-function.

1. Introduction

Lie algebras are among the basic tools of modern physics. Algebraic approaches to the problems of atomic, molecular, nuclear and hadronic physics have been extensively employed in the last decades, e.g. the interacting boson model in treating quadrupole collective motion [1], the description of the nuclear molecular states by dipole rotations and vibrations [2], and spectrum generating algebra techniques for scattering problems. Many applications have used the algebraic scattering theory [3, 4], in particular we mention their success in the classification of exactly solvable potentials [3, 5]. One-dimensional atom–molecule collisions have also been studied by using a combination of differential and algebraic techniques for a variety of potentials [6]. Another example is the recent introduction of supersymmetric quantum mechanics (SUSY), which allows pairing in isospectral potentials [7].

New directions appeared with the developing of the 'quantized' or q-deformed Lie algebras (quantum groups) [8], like exactly one-dimensional solvable potentials [9] or the analysis done in [10] (see also the references herein).

Such approaches opened the question of the geometrical interpretation of the algebraic Hamiltonian which is written as one of the Lie algebra operators or a bilinear combination in these operators. The geometrical-algebraic connection is well known for the Coulomb problem [11] and Morse or Pöschl–Teller potentials [3]. For the last case the algebraic interpretation is realized by taking an auxiliary coordinate. By using such an auxiliary

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coordinate, Kais and Levine [12] gave an algebraic interpretation of the states in the infinitely deep rectangular-well.

In the following we present an algebraic realization for the rectangular-well potential with special boundary conditions, namely boundary conditions that are useful to describe the scattering in the R-matrix framework [13, 14] without using an auxilliary coordinate. We also describe the problem of the oscillator potential on the real semiaxis in the algebraic framework. This approach will allow us to obtain algebraically a parametrization of the R-function [13] which contains all the information needed to describe the scattering. This parametrization could be considered as an extension of the algebraic scattering theory proposed by Iachello and co-workers [3, 4].

2. Rectangular-well potential with boundary conditions

We consider the problem of a particle in a one-dimensional rectangular-well potential $V(r) = -V_0$ if $r \in [0, a]$ where V_0 is the potential depth and a its radius. The eigenfunctions and the spectrum of the Hamiltonian $H_0 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - V_0$ with the boundary conditions

$$\begin{aligned} &\Phi(r=0) = 0 \\ &\frac{\mathrm{d}\Phi}{\mathrm{d}r}(r=a) = 0 \end{aligned} \tag{1}$$

are well known. They are given by

$$|\mu\rangle = \Phi_\mu(r) = \sqrt{\frac{2}{a}}\sin\left(\left(\mu + \frac{1}{4}\right)2\pi\frac{r}{a}\right) \tag{2}$$

$$E_\mu = \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2\left(\mu + \frac{1}{4}\right)^2 - V_0 \tag{3}$$

where $\mu \in \mathcal{Z}$. The functions Φ_μ are normalized: $\int_0^a \Phi^*_{\mu_1}(r)\Phi_{\mu_2}(r)\,\mathrm{d}r = \delta_{\mu_1\mu_2}$. We would like to note that the second relation in equation (1) represents a particular case of the general 'R-matrix boundary condition' of the form $\Phi'(a)/\Phi(a) =$ constant, used in the usual R-matrix approaches [13]. When this constant is very small or 0, the internal eigenstates can be interpreted physically as virtual levels [15]. In fact, the zero value of the first derivative of the wavefunction at $r = a$ implies the fact that the current of probability is zero, as in the case when the wavefunction is chosen to be zero.

We will now consider the problem of the R-matrix description of the scattering problem for this potential [13]. In this approach, the dynamics in the internal region is taken into account through the R-matrix. For the potential scattering, the R-matrix reduces to the R-function which can be written in terms of the internal spectrum and the values of the normalized eigenfunctions at the boundary $r = a$

$$R(E) = \sum_\mu \frac{\gamma_\mu^2}{E_\mu - E} \tag{4}$$

where $\gamma_\mu = \sqrt{\hbar^2/2ma}\,\Phi_\mu(a)$ and γ_μ^2 is the reduced width of the level μ. The R-function together with the logarithmic derivative of the outgoing wavefunction at $r = a$ allows a complete description of the scattering.

Our aim is to give an algebraic description of the eigenvalue problem for the Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - V_0 \tag{5}$$

with the R-matrix boundary condition (1).

We introduce the operator T, defined by its action: $Tf(r) = f(a-r)$. The corresponding differential realization of T is given by: $T = \mathrm{e}^{-a\partial_r}\mathrm{e}^{\mathrm{i}\pi r\partial_r}$. Consequently we introduce the operators:

$$\begin{aligned} J_z &= \frac{a}{2\pi}T\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{4} \\ J_\pm &= \left(\cos\left(2\pi\frac{r}{a}\right) \pm \sin\left(2\pi\frac{r}{a}\right)T\right)\left(\frac{a}{2\pi}T\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{4} \pm \frac{1}{2}\right) \end{aligned} \tag{6}$$

which act on the space of complex valued functions $L = \{f : [0,a] \to C | f \in L^2([0,a]),\ f \in C^1([0,a]),\ f(0) = 0\}$. Taking into account the definition of the operators J_z, $J_\pm$ and the obvious relations

$$\begin{aligned} T\frac{\mathrm{d}}{\mathrm{d}r} &= -\frac{\mathrm{d}}{\mathrm{d}r}T \\ T\cos\left(2\pi\frac{r}{a}\right) &= \cos\left(2\pi\frac{r}{a}\right)T \\ T\sin\left(2\pi\frac{r}{a}\right) &= -\sin\left(2\pi\frac{r}{a}\right)T \end{aligned} \tag{7}$$

we can establish the following commutation relations

$$\begin{aligned} [J_z, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= -2J_z. \end{aligned} \tag{8}$$

Also, we have the Hermiticity properties

$$J_z^\dagger = J_z \tag{9}$$

$$J_+^\dagger = J_-. \tag{10}$$

As an example, we prove the relation (9). We have

$$\begin{aligned} (f_1, (J_z + \tfrac{1}{4})f_2) &= -\int_0^a f_1^* \frac{a}{2\pi}\frac{\mathrm{d}f_2(a-r)}{\mathrm{d}r}\,\mathrm{d}r \\ &= -\frac{a}{2\pi}f_1^*(r)f_2(a-r)|_0^a + \int_0^a \left(\frac{a}{2\pi}\frac{\mathrm{d}f_1(r)}{\mathrm{d}r}\right)^* f_2(a-r)\,\mathrm{d}r \\ &= \int_0^a \left(\frac{a}{2\pi}\frac{\mathrm{d}f_1(a-r)}{\mathrm{d}(a-r)}\right)^* f_2(r)\,\mathrm{d}r \\ &= ((J_z + \tfrac{1}{4})f_1, f_2) \end{aligned}$$

where we have used the condition $f_{1,2}(r=0) = 0$. Relations (9) and (10) are now obvious. Therefore, the operators J_z, $J_\pm$ are closed under commutation relations (8) and express a realization of the Lie algebra $su(1,1)$. It is a straightforward exercise to prove that the action of the operators (6) on the eigenfunctions (2) can be written

$$\begin{aligned} J_z|\mu\rangle &= \mu|\mu\rangle \\ J_\pm|\mu\rangle &= (\mu \pm \tfrac{1}{2})|\mu \pm 1\rangle. \end{aligned} \tag{11}$$

The $su(1,1)$ Casimir operator is $C = J_z^2 - J_z - J_+J_-$, and we have

$$C|\mu\rangle = -\tfrac{1}{4}|\mu\rangle. \tag{12}$$

Relations (11) and (12) mean that the internal eigenstates $|\mu\rangle$ can be taken as the standard basis of the irreducible representation of $su(1,1)$ with $j = -\frac{1}{2}$ or $j(j+1) = -\frac{1}{4}$ and

$\mu \in \mathcal{Z}$. In fact, the relations (11) are the $j = -\frac{1}{2}$ case of the generator action on the standard basis.

Now, we can write the Hamiltonian (5) in the above algebraic framework as

$$H_0 = \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2\left(J_z + \frac{1}{4}\right)^2 - V_0. \tag{13}$$

Thus, the $su(1,1)$ Lie algebra is the dynamical algebra for the internal motion. The appropriate realization is given in terms of a single variable and it is a differential realization, i.e.

$$T = \sum_0^\infty \frac{(a-2r)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}r^n}$$

is an infinite-order differential operator.

The next step is to obtain a purely algebraic relation for the reduced widths for the above case. Since the reduced widths are proportional to the values of the wavefunction at the boundary, it seems that a purely algebraic description is not possible. Therefore, by taking realization (6) into account, we obtain the asymptotic connection:

$$\lim_{r\to a} J_\pm\varphi(r) = \lim_{r\to a}(J_z \pm \tfrac{1}{2})\varphi(r) \tag{14}$$

where $\varphi(r)$ is an arbitrary function of class C^1. If $\varphi(r) = \Phi_\mu(r)$ and taking into account equation (11), we have a simple recurrence relation:

$$(\mu \pm \tfrac{1}{2})\Phi_{\mu\pm1}(a) = (\mu \pm \tfrac{1}{2})\Phi_\mu(a). \tag{15}$$

Consequently the reduced widths do not depend on the quantum number μ as we have $\mu \in \mathcal{Z}$. Therefore, all the γ_μ are equal, and we can choose a certain one, e.g. γ_0. Then we can algebraically obtain the R-function up to a multiplicative factor in the form:

$$\begin{aligned} R(E) &= \sum_{\mu\in\mathcal{Z}} \gamma_0^2 \Bigg/ \left(\frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2\left(\mu + \frac{1}{4}\right)^2 - V_0 - E\right) \\ &= \frac{a\gamma_0^2}{\hbar}\sqrt{\frac{m}{2(V_0+E)}}\tan\left(\frac{a}{\hbar}\sqrt{2m(V_0+E)}\right). \end{aligned}$$

The coefficient γ_0 can be given only by the analytical expression of the state $|0\rangle$. It yields $\gamma_0^2 = \hbar^2/(ma^2)$.

So far we have been interested in the simplest problem of a rectangular potential. In the following we shall extend the algebraic description to other potentials.

3. Algebraization of other potentials

The realization (6) allows us to write

$$\begin{aligned} \cos\left(2\pi\frac{r}{a}\right) &= (2J_z - 1)^{-1}J_+ + (2J_z + 1)^{-1}J_- \\ &= J_+(2J_z + 1)^{-1} + J_-(2J_z - 1)^{-1}. \end{aligned} \tag{16}$$

It is important to note that expression (16) is not in the $su(1,1)$ universal enveloping algebra, but it can be considered to be well defined on a dense subspace of the $su(1,1)$ representation Hilbert space [16]. We can algebraically write $\cos(n2\pi r/a)$, $n \in N$, by using the expression of $\cos nx$ as an nth-order polynomial in $\cos x$. For example, using the particular value of the Casimir operator, we have

$$\cos\left(4\pi\frac{r}{a}\right) = 2(2J_z-1)^{-1}(2J_z-3)^{-1}J_+^2 + 2(2J_z+1)^{-1}(2J_z+3)^{-1}J_-^2. \tag{17}$$

In order to write algebraically a certain potential which can be written as a Fourier series we also need an expression for $\sin(2\pi r/a)$ in terms of the operators J_z, $J_\pm$. Such an expression seems to be a very complicated one, and we succeeded in obtaining it using the explicit realization of the standard basis (2). One can write

$$\begin{aligned}\sin\left(2\pi\frac{r}{a}\right) = \frac{1}{\pi}\left(2J_z+\frac{3}{2}\right)^{-1}\left(2J_z-\frac{1}{2}\right)^{-1} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(2J_z+\frac{3}{2}-n\right)^{-1}\left(2J_z-\frac{1}{2}-n\right)^{-1}\\ \times\left[\left(J_z-\frac{1}{2}\right)^{-1}J_+\right]^n + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(2J_z+\frac{3}{2}+n\right)^{-1}\left(2J_z-\frac{1}{2}+n\right)^{-1}\\ \times\left[\left(J_z+\frac{1}{2}\right)^{-1}J_-\right]^n.\end{aligned} \tag{18}$$

Therefore in principle, every potential developed in a Fourier series can be written in an algebraic form. As an example we shall study the Hamiltonian

$$\begin{aligned}H &= H_0 + 2\alpha\cos\left(2\pi\frac{r}{a}\right)\\ &= -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - V_0 + 2\alpha\cos\left(2\pi\frac{r}{a}\right)\\ &= \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2(J_z+\tfrac{1}{4})^2 - V_0 + 2\alpha\{(2J_z-1)^{-1}J_+ + (2J_z+1)^{-1}J_-\}\end{aligned} \tag{19}$$

where α is a coupling parameter. We would like to obtain the eigenstates of the algebraic Hamiltonian (19) in terms of the H_0 eigenstates, i.e. in terms of the $su(1,1)$ standard basis (11).

We suppose that the eigenstate $|E\rangle$ of H has the eigenvalue of energy E, $H|E\rangle = E|E\rangle$. We can write

$$|E\rangle = \sum_\mu C_\mu^E|\mu\rangle. \tag{20}$$

We admit that the action of H on the above sum commutes with the sum. When $\alpha = 0$, this assumption gives

$$\left(E - \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2\left(\mu+\frac{1}{4}\right)^2 + V_0\right)C_\mu^E = 0$$

for every integer μ, and one obtains that the energy E must be E_μ. Therefore, we have reobtained the H_0 spectrum and its eigenstates.

In the above assumption concerning the Hamiltonian (19) we obtain the following recursion relation

$$\alpha C_{\mu-1}^E + \left[\frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2\left(\mu+\frac{1}{4}\right)^2 - (E+V_0)\right]C_\mu^E + \alpha C_{\mu+1}^E = 0 \qquad \mu\in\mathcal{Z}. \tag{21}$$

By writing $C^E_{\mu+1} = \xi^E_\mu C^E_\mu$, relation (21) reads as

$$\alpha \frac{1}{\xi^E_{\mu-1}} + \left[\frac{\hbar^2}{2m} \left(\frac{2\pi}{a} \right)^2 \left(\mu + \frac{1}{4} \right)^2 - (E + V_0) \right] + \alpha \xi^E_\mu = 0. \tag{22}$$

For the convergence of series (20) at the boundary, where $|\mu\rangle \to$ constant, it is necessary to have $\lim_{\mu\to\infty} \xi_\mu = 0$ and $\lim_{\mu\to-\infty} 1/\xi_\mu = 0$. From equation (22), viewed as a defining relation for ξ^E_μ, we can obtain an equation for the unknown eigenvalues, involving continuous fractions. The lowest non-trivial order of approximation for these continuous fractions is equivalent to the truncation of the Hilbert space to the first three unperturbed states. It is easy to check that in the second order in α^2 the energy values contain terms corresponding to an order higher than two in the perturbation theory. Following such calculations and by introducing the energy eigenvalues into equation (21), one can obtain the normalized eigenstates.

More generally, if H contains, for example, a term $\cos(4\pi r/a)$ in the potential, then a recursion relation similar to (21) can be written connecting five coefficients C^E_μ.

4. Oscillator potential with a boundary condition in the origin

In the following, we want to show that the above procedure also works for noncompact intervals for the independent variable. As an example, we shall study the eigenfunction problem of the harmonic oscillator Hamiltonian on the real semi-axis with the R-matrix boundary condition

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x}(x = 0) = 0. \tag{23}$$

We take $m = \frac{1}{2}$, $\omega = 2$ and

$$H_0 = p^2 + x^2 = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 \qquad x \in [0, \infty). \tag{24}$$

The eigenfunctions of the above Hamiltonian with the boundary condition (23) are

$$|k\rangle = \Psi_k(x) = \frac{1}{\sqrt{\sqrt{\pi\hbar}2^{2k-1}(2k)!}} \exp\left(-\frac{1}{2\hbar} x^2 \right) H_{2k}\left(\sqrt{\frac{1}{\hbar}} x \right) \tag{25}$$

where $k = 0, 1, 2, \ldots$ and H_n are the Hermite polynomials. The corresponding energies are

$$E_k = 4\hbar(k + \tfrac{1}{4}). \tag{26}$$

In the following we realize the $su(1, 1)$ Lie algebra in terms of creation and annihilation operators. The Hamiltonian of the harmonic oscillator can be written in terms of the creation and annihilation operators a_+ and a_-:

$$a_\pm = \frac{1}{\sqrt{2\hbar}} (x \pm \mathrm{i} p) \tag{27}$$

which satisfy the commutation relation

$$[a_-, a_+] = 1. \tag{28}$$

The operators defined as

$$\begin{aligned} J_z &= \tfrac{1}{4}(a_+a_- + a_-a_+) \\ J_\pm &= \tfrac{1}{2} a^2_\pm \end{aligned} \tag{29}$$

satisfy the $su(1,1)$ commutation relations of $su(1,1)$ (see equation (8)). We also have the Hermiticity properties $J_z^\dagger = J_z$ and $J_+^\dagger = J_-$. For realization (29) we have the Casimir operator

$$C = J_z^2 - J_z - J_+J_- = -\tfrac{3}{16}$$

and $H = 4\hbar J_z$.

Therefore, the eigenfunctions (25) represent the standard basis of the $su(1,1)$ representation with $j = -\frac{1}{4}$. This representation belongs to the complementary series of representation, and the J_z spectrum is $m = \frac{1}{4} + k$, $k = 0, 1, 2, \ldots$. We have the action of the J_z, $J_\pm$ operators on the J_z eigenfunctions

$$\begin{aligned} &J_z|m\rangle = m|m\rangle \\ &J_\pm|m\rangle = \sqrt{(m \pm \tfrac{1}{4})(m \pm \tfrac{3}{4})}|m \pm 1\rangle. \end{aligned} \tag{30}$$

As in the rectangular-well potential case, using realization (29) for the $su(1,1)$ Lie algebra operators, we can write the asymptotic connection

$$\lim_{x\to 0} J_+ f(x) = -\lim_{x\to 0}(J_z + \tfrac{1}{4}) f(x). \tag{31}$$

In the coordinate realization, $|m\rangle = f(x)$, by taking into account relation (31), we obtain the recursion relation

$$\lim_{x\to 0} |m+1\rangle = -\sqrt{\frac{m+\frac{1}{4}}{m+\frac{3}{4}}} \lim_{x\to 0} |m\rangle. \tag{32}$$

The above recursion relation can be solved. It yields

$$(\lim_{x\to 0} |m\rangle)^2 = \frac{\Gamma\left(m+\frac{1}{4}\right)}{\Gamma\left(m+\frac{3}{4}\right)} \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} \left(\lim_{x\to 0}\left|\frac{1}{4}\right\rangle\right)^2. \tag{33}$$

We define the R-function [13] as

$$R(E) = \sum_m \frac{\gamma_m^2}{E_m - E} \tag{34}$$

where $\gamma_m^2 = \hbar^2(\lim_{x\to 0} |m\rangle)^2$. Therefore, we obtain

$$\gamma_m^2 = \frac{\Gamma\left(m+\frac{1}{4}\right)}{\Gamma\left(m+\frac{3}{4}\right)} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \gamma_{1/4}^2 \tag{35}$$

and the above defined R-function is

$$R(E) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\frac{1}{2}\right)/\,\Gamma(k+1)}{4\hbar\left(k+\frac{1}{4}\right) - E} \gamma_{1/4}^2 \tag{36}$$

where $\gamma_{1/4}$ cannot be fixed by algebraic means as in the previous rectangular-well case.

5. Other algebraic Hamiltonians

We have proved the connection between the $su(1,1)$ Lie algebra and the harmonic oscillator with boundary condition (23). The appropriate operator realization is constructed as second-order differential operators (see equation (29)). In the following we study an extension of the algebraic treatment for other potentials defined on the real semi-axis. The key point in the following is the relation

$$J_+ + 2J_z + J_- = x^2/\hbar \tag{37}$$

which can be obtained from equations (27) and (29).

Therefore, using relation (37), we can algebraically write all potentials defined on the positive semi-axis which can be developed in a Taylor series in x^2 convergent on the entire semi-axis. To illustrate the algebraic treatment for such a potential, let us take the simple example

$$\begin{aligned} H &= H_0 + \alpha x^2 = p^2 + (1+\alpha)x^2 \\ &= 4\hbar J_z + \hbar\alpha(J_+ + 2J_z + J_-). \end{aligned} \tag{38}$$

The Hamiltonian (38) represents a harmonic oscillator with frequency $\omega = 2\sqrt{1+\alpha}$. We can compare the results obtained for the spectrum of this Hamiltonian with the well known spectrum of the harmonic oscillator. We follow the same method that we used in the case of the rectangular-well potential. Thus, we obtain the recursion relation

$$\alpha\sqrt{k(k-\tfrac{1}{2})}C^E_{k-1} + \left[(4k+1)\left(1+\frac{1}{2}\alpha\right) - \epsilon\right]C^E_k + \alpha\sqrt{\left(k+\frac{1}{2}\right)(k+1)}C^E_{k+1} = 0 \tag{39}$$

where $|E\rangle = \sum_{k=0}^{\infty} C^E_k |k\rangle$ and $E = \hbar\epsilon$.

By using again the relation $C^E_{k+1} = \xi^E_k C^E_k$ for $k = 0, 1, 2, \ldots$ we obtain recurrence relations for the coefficients ξ^E_k:

$$\xi^E_0 = -\frac{\sqrt{2}}{\alpha}\left[\left(1+\frac{\alpha}{2}\right) - \epsilon\right]$$

$$\alpha\sqrt{k\left(k-\frac{1}{2}\right)}\frac{1}{\xi^E_{k-1}} + \left[(4k+1)\left(1+\frac{\alpha}{2}\right) - \epsilon\right] + \alpha\sqrt{\left(k+\frac{1}{2}\right)(k+1)}\xi^E_k = 0 \tag{40}$$

for $k = 1, 2, \ldots$. Using the above relations we obtain an equation for the unknown energy $E = \hbar\epsilon$. The approximation for the energy reads in the non-trivial lowest order:

$$\epsilon_0 = \left(1+\frac{\alpha}{2}\right) - \frac{\alpha^2}{8} \qquad \epsilon_1 = 5\left(1+\frac{\alpha}{2}\right) + \frac{\alpha^2}{8}. \tag{41}$$

These values can be compared with the exact values of the oscillator with $\omega = 2\sqrt{1+\alpha}$. This approach can also be used to obtain the values of the perturbed wavefunctions at $x = 0$ and, therefore, to calculate the R-function by the same method we used for the rectangular-well potential in section 2.

6. Conclusions

We have described two particular one-dimensional problems in an algebraic framework, i.e. the rectangular-well and harmonic-oscillator potentials. We have calculated the corresponding energy spectra and the R-functions. The present realization of the $su(1,1)$ Lie algebra operators, equations (17), (18) and (37), allows us to write a large class of potentials in algebraic form, i.e. any potential defined on a bounded domain and expressed in its Fourier series or defined on the positive semi-axis and expressed in Taylor series in even powers of x only.

An example for a trigonometric-like potential is worked out. For potentials more complicated than the examples presented above, algebraic expressions in terms of $J_\pm$, J_0 can be obtained by generalizing equations (16) and (18). In this respect one can obtain an algebraic Hamiltonian and the full exact solutions of its dynamical symmetry for the above presented R-matrix boundary conditions. We have proved the possibility of parametrizing the R-function without using the explicit form of the wavefunction in a manner close to the spirit of algebraic scattering theory [3, 4]. The technique developed in section 3 allows the algebraic analysis and the exact solvability for any L_2, smooth potential defined on a bounded domain or any even potential defined on $\mathcal{R}^+$ in principle.

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